

Dynamical Gaussian state transfer with quantum error correcting architecture

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Transferring a quantum state of a light field to a memory is of particular importance. However, this transfer is usually hampered because the memory system is subjected to some noise and this can limit the performance of the state transfer to a great extent. In this paper, we consider the transfer of a Gaussian state of light to a linear medium memory such as an opto-mechanical oscillator and propose a dynamical feedback controller that suppresses the noise in the memory system. To protect an unknown state, the feedback scheme employs the specific configuration of the quantum error correction; that is, a three-mode Gaussian state having appropriate syndromes is taken as the input. Correspondingly, the memory consists of three independent linear systems. The syndrome errors are estimated continuously in time through the measurement of the output field, and the results are then fed back to control the system. Because the input is Gaussian and the systems are all linear, it is possible to formulate the problem using the framework of the celebrated classical Kalman filtering and linear quadratic Gaussian control. A numerical simulation demonstrates the effectiveness of the control scheme.

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I. INTRODUCTION

Transferring a quantum state of light to a memory is of particular importance for various purposes in quantum information technologies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Candidates for a memory are largely divided into two categories: discrete variable systems such as an atom with distinct energy levels [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and continuous variable systems such as an opto-mechanical oscillator with a vibration mode [14, 15, 16, 17, 18, 19, 20, 21, 22]. Remarkably, some experimental demonstrations of quantum state transfer have been reported [1, 8, 16, 18, 19].

In reality, however, the memory performance is essentially limited by environmental noise that inevitably occurs during the transfer process. In a real experiment several state-of-the-art techniques should be, at least implicitly, employed to suppress such noise. However, to the best of our knowledge there exists no systematic method for suppressing the noise. Therefore, control theory for quantum memory needs to be explored.

In working out this subject, a key fact is that due to the dynamical noise the information contained in the memory state is gradually erased, rather than that the noise suddenly vanishes it. Hence, if such an unwanted change of state can be monitored continuously in time, then the measurement result could be fed back to suppress the noise. In fact, in [2], the probe light field for state transfer is continuously measured at the terminal of the optical path to indirectly detect the error during the transfer process. In this paper, we propose a dynamical control scheme that uses such continuous measurement results not only for the error detection but further for

feedback control to obtain the better performance of the state transfer. Fortunately, measurement-based quantum feedback control theory is well developed [23, 24, 25] and has many practical uses [26, 27, 28, 29, 30, 31, 32]; hence, it will be also useful for our scheme.

Here we describe two specific features of the proposed system-controller configuration. The first one is that the memory is a linear medium system such as a collective atomic ensemble or an opto-mechanical oscillator. In addition, the input state to be transferred is assumed to be a Gaussian state of light [33], in which case the memory state also becomes Gaussian. Then, the feedback control scheme has the form of celebrated *linear quadratic Gaussian (LQG)* control with *Kalman filtering* [24, 26, 29, 30, 31]. The second feature is that the controller employs the schematic of quantum error correction (QEC) for the purpose of protecting an *unknown* input state. The basic idea of QEC is to encode an unknown state into an enlarged codespace so that an appropriate *syndrome* measurement yields sufficient information about the error, which can then be used to correct the error [34, 35, 36, 37, 38, 39]. Use of this QEC strategy for quantum memory means that the input is a three-mode unknown Gaussian state having appropriate syndromes, and it is transferred to the memory that is correspondingly enlarged; during the transfer process, the output light fields are measured continuously in time, yielding the estimate of the syndrome errors that can be fed back to control the system.

The effectiveness of this control scheme is actually expected from the fact that the dynamical feedback control has successfully been applied to some QEC problems [40, 41, 42, 43, 44, 45, 46]. It should be noted that the problem considered in this paper is not a QEC problem itself; in QEC, the system state can be protected with the use of ancillary apparatus, whereas in our formalism, the initial state of the memory system cannot be protected, but rather the input state to be transferred to

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the system is what should be protected. This observation is vital in the sense that our scheme does not violate the no-go theorem for Gaussian QEC [47].

The paper is organized as follows. In Section II, as a preliminary, we describe a single-mode Gaussian state transfer. The dynamical control scheme is addressed in two sections. First, the state preparation and transfer processes are given in Section III. Section IV explains the error detection scheme (Kalman filter) and the feedback controller (LQG controller). Finally, in Section V, we numerically demonstrate the efficiency of our method by taking, as an example of the memory, an opto-mechanical oscillator manipulated under ultra-low temperature.

II. SINGLE-MODE GAUSSIAN STATE TRANSFER

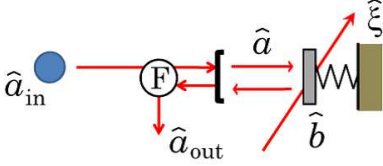


FIG. 1: (Color online) Opto-mechanical oscillator as an example of the single-mode linear quantum memory. The right end-mirror of the Fabry-Perot cavity serves as the oscillator, where \hat{a} and \hat{b} are annihilation operators corresponding to the intra-cavity and oscillator modes, respectively. Adiabatically eliminating the cavity mode \hat{a} , Eq. (1) is obtained. F denotes a Faraday isolator that facilitates one-way coupling between the input and cavity modes.

As a preliminary, this section is devoted to describe a single-mode Gaussian state transfer from a light field to a memory, where the former is a coherent CW laser field and the latter is served by a linear system such as an opto-mechanical oscillator shown in Fig. 1 (see e.g., [14, 17, 48] for a detailed discussion). The input is a coherent state $|\alpha_{\text{in}}\rangle$, with \hat{a}_{in} the annihilation operator of this coherent light field. The reflected output mode \hat{a}_{out} could be used to correct the errors occurring during the transfer process. In what follows we describe the dynamics of the memory system and show how much the input state $|\alpha_{\text{in}}\rangle$ is transferred to the memory.

A general single-mode open linear dynamics is described by the following quantum Langevin equation:

$$\frac{d\hat{b}}{dt} = -\frac{\nu + \Gamma}{2}\hat{b} - \sqrt{\nu}(\alpha_{\text{in}} + \hat{a}_0) - \sqrt{\Gamma}\hat{\xi}, \quad (1)$$

where \hat{b} denotes the system annihilation operator. Here, ν represents the coupling constant between the system and the input coherent light field $\hat{a}_{\text{in}} = \alpha_{\text{in}} + \hat{a}_0$, where α_{in} is the mean and \hat{a}_0 is the field annihilation operator. Moreover, we assume that the system couples with an unwanted single-mode noisy environment represented by

the annihilation operator $\hat{\xi}$ with mean zero. Γ is the coupling strength.

Let us now take the white noise approximation on the outer field modes \hat{a}_0 and $\hat{\xi}$; i.e., $\langle \hat{a}_0(t)\hat{a}_0^\dagger(t') \rangle = \delta(t - t')$ and $\langle \hat{\xi}(t)\hat{\xi}^\dagger(t') \rangle = (n + 1)\delta(t - t')$, where $n > 0$ represents the strength of the noise. In the case of thermal noise, n is the averaged photon number. This approximation allows us to represent the dynamics of \hat{b} in terms of the Ito-type quantum stochastic differential equation (QSDE) [49, 50]:

$$d\hat{b} = -\frac{\nu + \Gamma}{2}\hat{b}dt - \sqrt{\nu}\alpha_{\text{in}}dt - \sqrt{\nu}d\hat{A}_0 - \sqrt{\Gamma}d\hat{\Xi}, \quad (2)$$

where $\hat{A}_0(t) = \int_0^t \hat{a}_0(s)ds$ and $\hat{\Xi}(t) = \int_0^t \hat{\xi}(s)ds$ are the so-called quantum Wiener processes. Their infinitesimal increments satisfy the following quantum Ito-rule:

$$\begin{aligned} d\hat{A}_0 d\hat{A}_0^\dagger &= dt, \quad (d\hat{A}_0)^2 = d\hat{A}_0^{\dagger 2} = d\hat{A}_0^\dagger d\hat{A}_0 = 0, \\ d\hat{\Xi} d\hat{\Xi}^\dagger &= (n + 1)dt, \quad d\hat{\Xi}^\dagger d\hat{\Xi} = ndt, \quad d\hat{\Xi}^2 = d\hat{\Xi}^{\dagger 2} = 0. \end{aligned}$$

In this QSDE representation the input mode is written by $d\hat{A}_{\text{in}} = \alpha_{\text{in}}dt + d\hat{A}_0$. Since the field state is the vacuum, we have $\langle d\hat{A}_{\text{in}} \rangle = \alpha_{\text{in}}dt$. Also note that $\langle d\hat{\Xi} \rangle = 0$.

Now, let us see the steady state of the memory system. Using the above Ito rule we readily obtain the ordinary differential equations of $\langle \hat{b}(t) \rangle$ and $\langle \hat{b}(t)^\dagger \hat{b}(t) \rangle$, which give $\langle \hat{b}(\infty) \rangle = -2\sqrt{\nu}\alpha_{\text{in}}/(\nu + \Gamma)$ and

$$\langle \Delta \hat{q}(\infty)^2 \rangle = \langle \Delta \hat{p}(\infty)^2 \rangle = \frac{1}{2} + \frac{\Gamma n}{\nu + \Gamma} > \frac{1}{2},$$

where $\hat{q} = (\hat{b} + \hat{b}^\dagger)/\sqrt{2}$ and $\hat{p} = (\hat{b} - \hat{b}^\dagger)/\sqrt{2}i$ denote the dimensionless position and momentum operators of the system, respectively. This clearly illustrates that in the long time limit the system certainly acquires the information of the field input state $|\alpha_{\text{in}}\rangle$ in the mean sense (note that ν and Γ are assumed to be known). However the fluctuation of the state could become much bigger than the vacuum fluctuation $1/2$; in this case the state wrote down to the memory is far away from pure and has very small overlap with the input state.

III. PREPARATION AND TRANSFER PROCESSES OF THE INPUT STATE

In this paper, we study the system shown in Fig. 2. First, in the left box, the input state, which is a three-mode Gaussian state of light, is prepared. This input state is transferred into the memory shown in the middle box in the figure, that correspondingly consists of three identical linear systems; in the figure, opto-mechanical oscillators are particularly depicted as the memory system. Finally, the output fields are measured, and the results are used for feedback control.

A notable feature of this transfer scheme is that the input state is allowed to contain some unknown parameters. More specifically, the input state is generated by

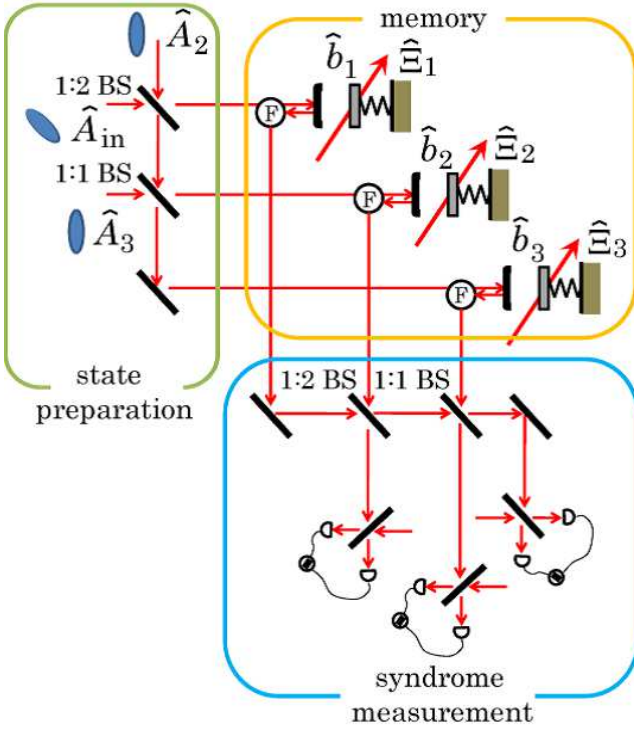


FIG. 2: (Color online) A schematic of the dynamical state transfer with QEC architecture, where in this picture an optomechanical oscillator is taken as a memory system.

embedding an unknown single-mode field \hat{A}_{in} into the three-mode light fields with the use of ancilla squeezed fields and beam splitters. This is no more than the schematic of QEC; actually, \hat{A}_{in} corresponds to a *source mode* to be protected, and for this purpose it is encoded into a larger Hilbert space where we can construct appropriate syndromes that do not depend on the unknown source mode but only contain information about the errors. Those errors could be corrected by feedback control through the syndrome measurement depicted in the lower box of the figure. Note that the three-mode encoding allows us to detect only either the position or the momentum error, but the proposed scheme can be extended to a nine-mode code to detect errors acting on both the position and momentum operators [39].

In this section, we describe the above-mentioned state preparation and transfer processes, and then evaluate in detail the memory performance, when without any error correction.

A. State preparation

For discrete variable systems, a series of CNOT gates is often used for encoding. A possible continuous-variable analogue is realized in linear optics circuit shown in the left box of Fig. 2; first, the information source mode \hat{A}_{in} is mixed via a 1 : 2 beam splitter with an ancilla squeezed

vacuum field denoted by \hat{A}_2 , and one of the outputs is further combined with the second ancilla squeezed vacuum field \hat{A}_3 . This combination of beam splitters is called the *tritter* [39].

Let us describe the above encoding process in detail. As in the single-mode case discussed in Sec. II, we write the source mode as $d\hat{A}_{in} = \alpha_{in}dt + d\hat{A}_1$, where α_{in} and \hat{A}_1 are the mean amplitude and the quantum fluctuation, respectively. In particular, we assume

$$\alpha_{in} \in \mathbb{R},$$

implying that the momentum element of the mean is known to be zero. As mentioned above, in this case, the three-mode encoding is sufficient to detect the position errors. The quantum fluctuation \hat{A}_j ($j = 1, 2, 3$) is a stochastic process satisfying $\langle d\hat{A}_j \rangle = 0$ and the following *quantum Ito rule*:

$$\begin{aligned} d\hat{A}_j d\hat{A}_j^\dagger &= (N_j + 1)dt, & d\hat{A}_j^\dagger d\hat{A}_j &= N_j dt, \\ d\hat{A}_j^2 &= M_j dt, & d\hat{A}_j^{\dagger 2} &= M_j^* dt, \end{aligned}$$

where the parameters $N_j \geq 0$ and $M_j \in \mathbb{C}$ have to satisfy $N_j(N_j + 1) \geq |M_j|^2$. The fluctuation parameters N_1 and M_1 of the source mode are unknown. For the ancilla modes \hat{A}_2 and \hat{A}_3 , we set

$$M_2 = M_3 = \frac{e^\mu - e^{-\mu}}{4}, \quad N_2 = N_3 = \frac{e^\mu + e^{-\mu} - 2}{4},$$

which satisfy $N_j(N_j + 1) = M_j^2$; that is, the ancilla modes are the identical pure squeezed vacuum fields with squeezing parameter μ . Let us collect the field quadratures in a single vector as

$$\hat{W}_1 = (\hat{Q}_1, \hat{P}_1, \hat{Q}_2, \hat{P}_2, \hat{Q}_3, \hat{P}_3)^\top, \quad (3)$$

where $\hat{Q}_j = (\hat{A}_j + \hat{A}_j^\dagger)/\sqrt{2}$ and $\hat{P}_j = (\hat{A}_j - \hat{A}_j^\dagger)/\sqrt{2}i$. Then, through the tritter, \hat{W}_1 becomes $d\hat{W}'_1 = \beta dt + T d\hat{W}_1$, where T is the orthogonal matrix corresponding to the combination of the above-mentioned 1 : 2 and half beam splitters:

$$T = \begin{pmatrix} \sqrt{1/3} & 0 & -\sqrt{2/3} & 0 & 0 & 0 \\ 0 & \sqrt{1/3} & 0 & -\sqrt{2/3} & 0 & 0 \\ \sqrt{1/3} & 0 & \sqrt{1/6} & 0 & \sqrt{1/2} & 0 \\ 0 & \sqrt{1/3} & 0 & \sqrt{1/6} & 0 & \sqrt{1/2} \\ \sqrt{1/3} & 0 & \sqrt{1/6} & 0 & -\sqrt{1/2} & 0 \\ 0 & \sqrt{1/3} & 0 & \sqrt{1/6} & 0 & -\sqrt{1/2} \end{pmatrix},$$

and

$$\beta = \frac{\sqrt{2}\alpha_{in}}{\sqrt{3}}(1, 0, 1, 0, 1, 0)^\top.$$

Using the Ito rule, it is found that the position quadratures of \hat{W}'_1 satisfy

$$\frac{d}{dt} \langle (\hat{Q}'_i - \hat{Q}'_j)^2 \rangle = e^\mu, \quad \forall i \neq j. \quad (4)$$

This quantity is essentially equivalent to the power spectrum density of $\langle(\hat{Q}'_i - \hat{Q}'_j)^2\rangle$ at the center frequency of the carrier laser field [51]. We then find that Eq. (4) becomes zero for every i, j when taking the limit $\mu \rightarrow -\infty$. This means that, in the Schrodinger picture, this ideal input state generated through the tritter lives in the codespace spanned by $\{|Q, Q, Q\rangle\}$ and is of the following GHZ-like form :

$$|\tilde{\psi}_{\text{in}}\rangle = \int \psi(Q)|Q, Q, Q\rangle dQ, \quad (5)$$

where $\psi(Q)$ is a Gaussian wave function of the source state. The tilde means that the state is the ideal one.

Next let us focus on the following quantity:

$$\mathcal{P}_{\text{fd}} := \langle(\hat{Q}'_1 - \hat{Q}'_2)^2\rangle + \langle(\hat{Q}'_2 - \hat{Q}'_3)^2\rangle + \langle(\hat{Q}'_3 - \hat{Q}'_1)^2\rangle + 3\langle(\hat{P}'_1 + \hat{P}'_2 + \hat{P}'_3)^2\rangle, \quad (6)$$

which leads to

$$\frac{d\mathcal{P}_{\text{fd}}}{dt} = 3e^\mu + \frac{9}{2}(2N_1 - M_1 - M_1^* + 1).$$

If \hat{A}_1, \hat{A}_2 , and \hat{A}_3 are all coherent fields, we have $d\mathcal{P}_{\text{fd}}/dt = 7.5$, which thus can be interpreted as the classical limit. This means that a non-classical input state is generated when $d\mathcal{P}_{\text{fd}}/dt < 7.5$, which is now equivalent to that $\mu < 0$ and $M_1 \geq 0$. Moreover, it is known that a symmetric state such as the one taken here is entangled if $d\mathcal{P}_{\text{fd}}/dt < 6$ [52, 53]. For instance setting a coherent source state, i.e., $N_1 = M_1 = 0$, and taking the ideal limit $\mu \rightarrow -\infty$, we obtain $d\mathcal{P}_{\text{fd}}/dt \rightarrow 4.5$; hence in this case the GHZ-like state (5) is indeed entangled.

B. Transfer process

Let us next describe the transfer process of the mode \hat{W}'_1 to the memory served by the three identical linear systems. The dynamics of the memory is given by the combination of Eq. (2) with the input field replaced by $d\hat{W}'_1$; that is, the vector of quadratures,

$$\hat{x} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \hat{q}_3, \hat{p}_3)^\top$$

with $\hat{q}_j = (\hat{b}_j + \hat{b}_j^\dagger)/\sqrt{2}$ and $\hat{p}_j = (\hat{b}_j - \hat{b}_j^\dagger)/\sqrt{2}i$, satisfies the following QSDE:

$$\begin{aligned} d\hat{x} &= A\hat{x}dt + udt - \sqrt{\nu}d\hat{W}'_1 - \sqrt{\Gamma}d\hat{W}_2 \\ &= A\hat{x}dt + udt - \sqrt{\nu}\beta dt + Bd\hat{W}, \end{aligned} \quad (7)$$

where $\hat{W} = (\hat{W}_1, \hat{W}_2)^\top$ and

$$A = -\frac{\nu + \Gamma}{2}I_6, \quad B = (-\sqrt{\nu}T, -\sqrt{\Gamma}I_6).$$

The fluctuation vector \hat{W}_1 is given by Eq. (3) and

$$\hat{W}_2 = \sqrt{2}(\Re(\hat{\Xi}_1), \Im(\hat{\Xi}_1), \Re(\hat{\Xi}_2), \Im(\hat{\Xi}_2), \Re(\hat{\Xi}_3), \Im(\hat{\Xi}_3))^\top$$

is the vector of noise processes to which the linear systems are subjected; hence $\hat{\Xi}_j$ ($j = 1, 2, 3$) satisfies

$$d\hat{\Xi}_j d\hat{\Xi}_j^\dagger = (n+1)dt, \quad d\hat{\Xi}_j^\dagger d\hat{\Xi}_j = ndt, \quad d\hat{\Xi}_j^2 = d\hat{\Xi}_j^{2\dagger} = 0.$$

Again, $n > 0$ represents the noise strength. Finally, $u \in \mathbb{R}^6$ represents the control input for the memory. Note here it is assumed that both the quadratures of each linear system can be controlled. In what follows of this section we set $u = 0$.

The state transfer can be evaluated in terms of the mean vector and the covariance matrix, as discussed in Sec. II. First, the mean vector $\langle\hat{x}\rangle = (\langle\hat{q}_1\rangle, \dots, \langle\hat{p}_3\rangle)^\top$ obeys $d\langle\hat{x}\rangle/dt = A\langle\hat{x}\rangle - \sqrt{\nu}\beta$, thus in the long-time limit we have $\langle\hat{x}(\infty)\rangle = \sqrt{\nu}A^{-1}\beta = -2\sqrt{\nu}\beta/(\nu + \Gamma)$. That is, the system state becomes Gaussian with mean vector parallel to β , implying that the memory certainly acquires information of the input state in the mean sense (note that ν and Γ are assumed to be known). Next let us consider the covariance matrix of the system:

$$V = \langle\Delta\hat{x}\Delta\hat{x}^\top + (\Delta\hat{x}\Delta\hat{x}^\top)^\top\rangle/2, \quad \Delta\hat{x} = \hat{x} - \langle\hat{x}\rangle.$$

Using the Ito rule, the time evolution of V is found in the following form called the Lyapunov differential equation:

$$\dot{V} = AV + VA^\top + \Gamma(n+1/2)I_6 + \nu T\Lambda T^\top, \quad (8)$$

where

$$\begin{aligned} \Lambda &= \text{diag}\{\Lambda_1, \Lambda_2, \Lambda_3\}, \\ \Lambda_j &= \begin{pmatrix} N_j + \Re(M_j) + 1/2 & \Im(M_j) \\ \Im(M_j) & N_j - \Re(M_j) + 1/2 \end{pmatrix}. \end{aligned}$$

Note $\Lambda_2 = \Lambda_3$ by assumption. Now, to explicitly evaluate the state transfer, let us take the source state to be coherent, in which case $M_1 = N_1 = 0$; then the steady solution of Eq. (8) is obtained as

$$V_\infty = T\text{diag}\{V_1, V_2, V_3\}T^\top, \quad V_j = \text{diag}\{v_j^+, v_j^-\},$$

where

$$v_j^\pm = \frac{\nu(2N_j \pm 2M_j + 1) + \Gamma(1 + 2n)}{2(\nu + \Gamma)}.$$

Note $V_2 = V_3$. Then we have the explicit form of the fidelity between the input state $|\psi_{\text{in}}\rangle$ and the steady state of the memory with its mean appropriately displaced:

$$\begin{aligned} F &= \langle\psi_{\text{in}}|\hat{\rho}_\infty|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{\det(V_\infty + V_{\text{in}})}} \\ &= \prod_{\sigma=0, +\mu, -\mu} \frac{2(\nu + \Gamma)}{2\nu e^\sigma + \Gamma(e^\sigma + 1 + 2n)}, \end{aligned} \quad (9)$$

where $V_{\text{in}} = T\Lambda T^\top$ is the covariance matrix of the input state $|\psi_{\text{in}}\rangle$. Note that Eq. (5) is the ideal limit of $|\psi_{\text{in}}\rangle$. Eq. (9) shows that, when $\Gamma = 0$, we have $F = 1$ without respect to the value of μ . This means that the input state is perfectly transferred into the memory when it is

not coupled to the noisy environment. But in reality the system must be subjected to some noise; QEC is expected to overcome this issue. Motivated by some continuous variable QEC protocols found in the literature [36, 37, 38, 39], let us take the operator $\hat{q}_i - \hat{q}_j$ ($i, j = 1, 2, 3$) as a syndrome. Actually, this has desirable properties shown as follows. First, it can be seen from the structure of the system dynamics (7) that the syndrome does not contain the unknown parameters α_{in}, N_1 , and M_1 . Second, in the ideal limit $\mu \rightarrow -\infty$, the variance of each syndrome is given by

$$\langle \Delta(\hat{q}_i - \hat{q}_j)^2 \rangle = \frac{\Gamma(2n+1)}{(\nu + \Gamma)}, \quad \forall i \neq j, \quad (10)$$

which takes zero only when the system does not couple to the noisy environment. This means that, if in the small interval $[t, t + dt]$ only one of the three linear systems is subjected to the noise, simultaneous “measurement” of the syndrome operators can detect in which system that error has occurred and how much it is. (The reason why the double quotation is taken is that the syndrome operator cannot be directly measured in our setting, but it needs to be *estimated*; see the next section.) Because of these two properties, in the limit $\mu \rightarrow -\infty$, the operators $\hat{q}_i - \hat{q}_j$ certainly plays a role of a syndrome. Here it should be noted that F takes the maximum value when $\mu = 0$; that is, without error correction the encoding process merely degrades the performance of the state transfer [54]. Hence we really need the correction process.

Before closing this section, let us evaluate the entanglement of the system state, using essentially the same quantity as Eq. (6). Again in the case $N_1 = M_1 = 0$, we have

$$\begin{aligned} \mathcal{P}_{\text{sys}} &:= \langle \Delta(\hat{q}_1 - \hat{q}_2)^2 \rangle + \langle \Delta(\hat{q}_2 - \hat{q}_3)^2 \rangle + \langle \Delta(\hat{q}_3 - \hat{q}_1)^2 \rangle \\ &\quad + 3\langle \Delta(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)^2 \rangle \\ &= \frac{(4.5 + 3e^\mu)\nu}{\nu + \Gamma} + \frac{(7.5 + 15n)\Gamma}{\nu + \Gamma}. \end{aligned}$$

When $\Gamma = 0$ and $\mu \rightarrow -\infty$, it is found $\mathcal{P}_{\text{sys}} \rightarrow 4.5$. Hence, together with $\langle \Delta(\hat{q}_i - \hat{q}_j)^2 \rangle = 0$ in this case, the memory state becomes an entangled GHZ-like state. But in the realistic situation with $\Gamma > 0$ such entanglement can vanish. Particularly in the ideal case $\mu \rightarrow -\infty$ the sufficient condition for entanglement, $\mathcal{P}_{\text{sys}} < 6$, leads to $n < 0.1(\nu/\Gamma) - 0.1$; that is, in the case of thermal noise, the averaged photon number of the environment field must be less than about one order of magnitude below the S/N rate ν/Γ .

IV. SYNDROME FILTER AND DYNAMICAL FEEDBACK CONTROL

In the previous section it was found that $\hat{q}_i - \hat{q}_j$ serves as the syndrome through which the error can be detected. Since this observable cannot be measured directly in our

setting, it should be appropriately estimated through indirect measurement. In this section we first describe a continuous-time estimator, i.e., *the filter*, for the syndromes. We then present the dynamical feedback controller that is based on the syndrome filter.

A. Syndrome Filter

Let us focus on the following two kinds of operator vectors:

$$\hat{s}_1 = \tilde{B}_1 \hat{x} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2}(\hat{p}_1 + \hat{p}_2 + \hat{p}_3) \\ \hat{q}_2 + \hat{q}_3 - 2\hat{q}_1 \\ \sqrt{3}(\hat{q}_2 - \hat{q}_3) \end{pmatrix} \quad (11)$$

and

$$\hat{s}_2 = \tilde{B}_2 \hat{x} = \frac{1}{\sqrt{6}} \begin{pmatrix} \hat{q}_2 + \hat{q}_3 - 2\hat{q}_1 \\ \sqrt{3}(\hat{q}_2 - \hat{q}_3) \end{pmatrix}. \quad (12)$$

Here $\tilde{B}_1 := Z_1 T^\top$ and $\tilde{B}_2 := Z_2 T^\top$ are isometric matrices with

$$Z_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

If only one of the three linear systems is subjected to the noise, \hat{s}_2 can detect in which system that error has occurred. Hence, \hat{s}_2 serves as the syndrome operator. The first element of \hat{s}_1 is used to evaluate entanglement of the memory state.

Our task is to estimate the operator \hat{s}_1 or \hat{s}_2 through certain indirect measurement. The lower box in Fig. 2 depicts an optical configuration that achieves this goal for the case of \hat{s}_1 ; that is, we construct the tritter acting on the output fields that are reflected at the field-system coupler. The observables to be measured by homodyne detectors with appropriate LO phase are then given by

$$d\hat{Y}_1 = \sqrt{2\nu}\hat{s}_1 dt + \sqrt{2}(d\hat{P}_1, d\hat{Q}_2, d\hat{Q}_3)^\top. \quad (13)$$

Hence measuring \hat{Y}_1 actually implies the indirect measurement of \hat{s}_1 . Note this contains the source fluctuation $d\hat{P}_1$, hence in this case only the mean value of the input state, β , can be kept unknown. On the other hand, for the case of \hat{s}_2 , two of the three output fields are measured;

$$d\hat{Y}_2 = \sqrt{2\nu}\hat{s}_2 dt + \sqrt{2}(d\hat{Q}_2, d\hat{Q}_3)^\top. \quad (14)$$

\hat{Y}_2 does not contain both the mean and covariance of the source state, thus we can estimate the syndromes without respect to the unknown parameters. Note here that the position squeezing of the ancilla fields reduces the fluctuation $d\hat{Q}_2$ and $d\hat{Q}_3$; in particular taking the ideal limit $\mu \rightarrow -\infty$ we have $d\hat{Y}_2 = \sqrt{2\nu}\hat{s}_2 dt$, thus this is no more than the syndrome measurement.

We now describe the filter of \hat{s}_i , which is constructed with the measurement results of \hat{Y}_i ($i = 1, 2$). Let us write Eq. (13) or (14) in the following general form:

$$d\hat{Y} = C\hat{x}dt + Dd\hat{W}. \quad (15)$$

For the case of estimating \hat{s}_1 , the matrices correspond to $C = \sqrt{2\nu}\hat{B}_1 = \sqrt{2\nu}Z_1T^\top$ and $D = \sqrt{2}(Z_1, O_{3 \times 6})$. Also for the case of \hat{s}_2 , $C = \sqrt{2\nu}\hat{B}_2 = \sqrt{2\nu}Z_2T^\top$ and $D = \sqrt{2}(Z_2, O_{2 \times 6})$. For the simple notation, we do not put the index i on \hat{Y} , C , and D . The continuous measurement of \hat{Y} enables us to construct the filter for the dynamics (7):

$$d\pi(\hat{x}) = A\pi(\hat{x})dt + udt - \sqrt{\nu}\beta dt + Kd\bar{w}, \quad (16)$$

$$K = (V_c C^\top - \sqrt{2\nu}T\Lambda Z^\top)(2Z\Lambda Z^\top)^{-1}, \quad (17)$$

$$d\bar{w} = dy - C\pi(\hat{x})dt, \quad (18)$$

where Z implies Z_1 or Z_2 . The set of above equations is called the quantum Kalman filter [24, 26, 29, 30, 31]. The 6-dimensional c-number vector $\pi(\hat{x}_t) = \mathbb{E}(\hat{x}_t|\mathcal{Y}_t)$ represents the quantum conditional expectation of \hat{x}_t conditioned on the set of measurement data $\mathcal{Y}_t = \{y_s \mid 0 \leq s \leq t\}$ with y_s the measurement result of \hat{Y}_s . Note $\pi(\hat{x}_t)$ is the least mean square estimate of \hat{x}_t . V_c is the conditional covariance matrix $V_c = \pi(\Delta\hat{x}\Delta\hat{x}^\top + (\Delta\hat{x}\Delta\hat{x}^\top)^\top)/2$ with $\Delta\hat{x} = \hat{x} - \pi(\hat{x})$, and it obeys the following Riccati differential equation:

$$\begin{aligned} \dot{V}_c = & AV_c + V_c A^\top + \nu T\Lambda T^\top + \Gamma\left(n + \frac{1}{2}\right)I_6 \\ & - 2KZ\Lambda Z^\top K^\top, \end{aligned} \quad (19)$$

where K is given by Eq. (17). Note this is not a stochastic process. The filtering equation for the syndrome \hat{s}_1 or \hat{s}_2 can be directly obtained from Eq. (16) as follows:

$$\begin{aligned} d\pi(\hat{s}) &= d\tilde{B}\pi(\hat{x}) \\ &= \tilde{B}A\pi(\hat{x})dt + \tilde{B}udt - \sqrt{\nu}\tilde{B}\beta dt + \tilde{B}Kd\bar{w} \\ &= \tilde{A}\pi(\hat{s})dt + \tilde{B}udt + \tilde{K}d\bar{w}, \end{aligned} \quad (20)$$

where $\tilde{A} = \tilde{B}A\tilde{B}^\top$ and $\tilde{K} = \tilde{B}K$. Here, \tilde{B} implies \tilde{B}_1 or \tilde{B}_2 . Also we have dropped the index of \hat{s}_i and simply denote \hat{s} . The unknown parameter α_{in} does not appear in Eq. (20) due to $\tilde{B}\beta = 0$. Moreover, for the case of estimating \hat{s}_2 , the corresponding Kalman gain $\tilde{K} = \tilde{B}_2K$ does not contain N_1 and M_1 , thus the filter can update the estimate of \hat{s}_2 without respect to the unknown source state. We call Eq. (20) the *syndrome filter*.

B. LQG control for the state transfer

Due to the coupling to the noisy environment, the system state must escape from the codespace spanned by $\{|q, q, q\rangle\}$, but this error can be detected by estimating the syndrome operators. Therefore, it is expected that

a feedback controller minimizing the estimated value of the syndromes corrects that error. Note that the controller must be a dynamical one so that it can deal with the dynamical noise; hence, the minimization should be carried out through the whole transfer process. Fortunately, because of the linearity of both the dynamics and the output equation, this requirement is satisfied when employing the LQG control strategy [24, 26, 29, 30, 31]. That is, we can construct an optimal control law u^* that minimizes the following quadratic-type cost function:

$$J = \left\langle \frac{1}{2} \int_0^T (\hat{s}^\top Q \hat{s} + u^\top R u) dt \right\rangle, \quad (21)$$

where $R = R^\top > 0$ represents the penalty for the control input. Corresponding to the syndrome operators \hat{s}_1 or \hat{s}_2 , the weighting matrix Q is set to:

$$Q_1 = \text{diag}\{9, 3, 3\}, \quad Q_2 = \text{diag}\{3, 3\}.$$

It is immediately seen that

$$\begin{aligned} \hat{s}_1^\top Q_1 \hat{s}_1 &= (\hat{q}_1 - \hat{q}_2)^2 + (\hat{q}_2 - \hat{q}_3)^2 + (\hat{q}_3 - \hat{q}_1)^2 \\ &\quad + 3(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)^2, \\ \hat{s}_2^\top Q_2 \hat{s}_2 &= (\hat{q}_1 - \hat{q}_2)^2 + (\hat{q}_2 - \hat{q}_3)^2 + (\hat{q}_3 - \hat{q}_1)^2. \end{aligned} \quad (22)$$

Therefore, for both cases the optimal control tries to minimize the syndrome errors. In the case of \hat{s}_1 , the optimal control further takes into account the effect of entanglement and tries to keep the GHZ-like form of the state. The LQG control theory gives the explicit form of the (stationary) optimal controller:

$$u^* = F\pi(\hat{s}) = -R^{-1}\tilde{B}^\top P\pi(\hat{s}), \quad (23)$$

where P is the solution to the following algebraic Riccati equation:

$$\tilde{A}^\top P + P\tilde{A} + Q - P\tilde{B}R^{-1}\tilde{B}^\top P = 0. \quad (24)$$

Let us here assume that $R = rI_6 > 0$. Then, corresponding to \hat{s}_1 and \hat{s}_2 , the matrix P is respectively obtained in the following form:

$$P_1 = \text{diag}\{f_1, f_2, f_2\}, \quad P_2 = \text{diag}\{f_2, f_2\}, \quad (25)$$

where

$$\begin{aligned} f_1 &= -\frac{\nu + \Gamma}{2} + \sqrt{\frac{(\nu + \Gamma)^2}{4} + \frac{9}{r}}, \\ f_2 &= -\frac{\nu + \Gamma}{2} + \sqrt{\frac{(\nu + \Gamma)^2}{4} + \frac{3}{r}}. \end{aligned}$$

We here give two brief remarks. (i) The first one is about the structure of the optimal feedback control input u^* . In the case of \hat{s}_1 , it is given by

$$u^* = \frac{f_2}{3r} \begin{pmatrix} \pi(\hat{q}_2 - \hat{q}_1) + \pi(\hat{q}_3 - \hat{q}_1) \\ -f_1\pi(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)/f_2 \\ \pi(\hat{q}_1 - \hat{q}_2) + \pi(\hat{q}_3 - \hat{q}_2) \\ -f_1\pi(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)/f_2 \\ \pi(\hat{q}_1 - \hat{q}_3) + \pi(\hat{q}_2 - \hat{q}_3) \\ -f_1\pi(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)/f_2 \end{pmatrix}.$$

The first element can be regarded as a linearization to the following sign function ($\lambda = f_2/3r$):

$$\lambda[\text{sgn}(\pi(\hat{q}_2 - \hat{q}_1)) + \text{sgn}(\pi(\hat{q}_3 - \hat{q}_1))],$$

where $\text{sgn}(x) = +1$ if $x > 0$ and -1 otherwise. Hence, in the small interval $[t, t + dt)$, if the position shift of \hat{q}_1 due to the thermal noise is larger than the others, the controller adds the largest inverse shift $-2\lambda dt$ on $d\hat{q}_1$ so as to cancel out the error. The other elements of u^* have similar meanings. It should be maintained that the optimal controller obtained in the LQG control setup has this kind of natural structure found in the usual QEC scheme. (ii) The second remark is related to the no-go theorem [47] mentioned in Sec. I. The QEC problem corresponding to our setting is that the system's initial state is to be protected by feedback control. However, this goal is never accomplished, because, in general, a stable Kalman filter forgets the initial values of $\pi(\hat{x}_0)$ and $V_c(0)$; that is, the initial Gaussian state with mean $\pi(\hat{x}_0)$ and covariance $V_c(0)$ cannot be protected. This is a way to understand the no-go theorem for Gaussian QEC from the dynamical control viewpoint.

V. SIMULATION

In this section, a numerical simulation is provided to demonstrate effectiveness of the proposed control scheme. As an example of the memory we here take an optomechanical oscillator shown in FIG. 1. The oscillator must be subjected to thermal noise even at ultra-low temperature; in this case the noise strength n corresponds to the averaged photon number $n = (e^{\hbar\omega_m/k_B T} - 1)^{-1}$, where k_B , T , and ω_m denote the Boltzmann constant, the temperature, and the carrier frequency of the thermal channel, respectively.

The parameters are taken as follows. The oscillator couples with the input field with strength $\nu/2\pi = 30$ kHz, while for the thermal channel it is $\Gamma/2\pi = 1$ Hz. The thermal channel is with frequency $\omega_m/2\pi = 10$ MHz and with temperature $T = 4$ K, which leads to $n = 8.8 \times 10^3$. The mean value of the input state is taken as $\alpha_{\text{in}} = -230 \text{ Hz}^{1/2}$, which leads to $-\sqrt{\nu}\beta = [100, 0, 100, 0, 100, 0]$. Note that $|\alpha_{\text{in}}|^2$ is the mean photon number per unit time.

A. Coherent source state

First let us consider the case where the source state is taken as a coherent state $|\alpha_{\text{in}}\rangle$. This means that the input fluctuation is set to $N_1 = M_1 = 0$, implying that, in addition to the syndrome operators, $\hat{p}_1 + \hat{p}_2 + \hat{p}_3$ can be estimated. Hence, we can now apply the feedback control scheme based on $\pi(\hat{s}_1)$.

Figure 3 shows how much the encoding and the feedback control improve the fidelity between the three-mode

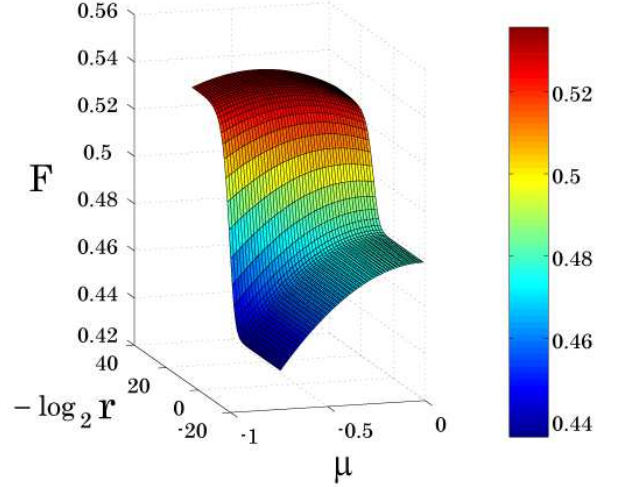


FIG. 3: (Color online) Fidelity between the input state and the steady state of the controlled oscillators, versus the squeezing parameter μ and the control strength $-\log_2 r$.

input state and the oscillators' steady state. They are quantified as follows. (i) Encoding strength is directly evaluated by the squeezing parameter μ ; actually, as indicated by Eq. (4), making $-\mu$ bigger means that the input state is going to approximate the GHZ-like state (5). (ii) The LQG optimal control input u^* can take possibly a bigger value by tuning the penalty parameter r smaller (Recall that we have set $R = rI_6$ in Eq. (21).) Hence, $-\log_2 r$ is interpreted as the control strength. (iii) The fidelity is given by the same form as Eq. (9) with V_∞ now replaced by the covariance matrix of the ensemble average of the *controlled* system state, say V'_∞ ; that is, $F = 1/\sqrt{\det(V'_\infty + V_{\text{in}})}$. Note that the controlled system variable \hat{x}_t stochastically changes in time according to the QSDE (7) with the input u^* a function of the estimate $\pi(\hat{s}_1)$, i.e., Eq. (23). Hence V'_∞ corresponds to the first 6×6 block matrix of the 9×9 covariance matrix of $(\hat{x}^\top, \pi(\hat{s}_1)^\top)^\top$ that satisfies the controlled Lyapunov equation (A1); see Appendix for the detailed calculation.

Let us now discuss the performance of the LQG optimal feedback control. It is observed from Fig. 3 that bigger control strength indicates bigger fidelity for all value of μ . This means that the feedback control can always reduce the excess fluctuation brought from the thermal environment, without respect to how much the source state is encoded. Regarding the encoding strength, however, care should be taken in its choice. Actually, while larger squeezing enables us to detect a bigger error signal which can induce more efficient feedback control, at the same time the memory state becomes more sensitive to the thermal noise. In other words, too much squeezing makes the input state fragile to the noise; consequently there exists an optimal value of the squeezing parameter, and it is about $\mu^* = -0.4$. Note this value can be

reached within the current technology. In this case, the amount of improvement of the fidelity via the feedback control compared to the case without encoding and control is about 0.05; but this is not a big improvement. One reason to this limitation is that the error taken in this paper is the worst one in the sense that each system is subjected to the thermal noise for all time; this kind of error is not fully tractable via the standard QEC protocol. Therefore, it is expected that the proposed control scheme could show possibly much better performance of the state transfer against some weak noise such as simple displacement acting only one of the oscillators.

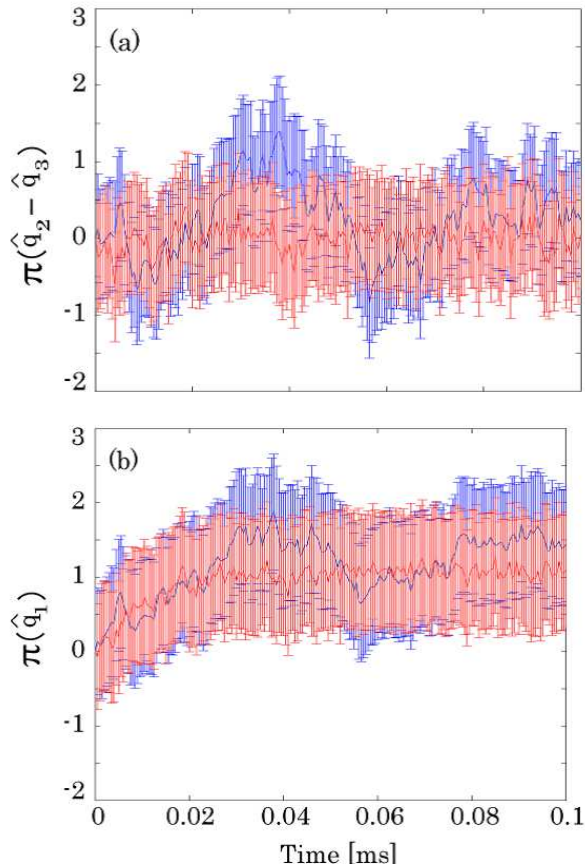


FIG. 4: (Color online) Time evolutions of (a) $\pi(\hat{q}_2 - \hat{q}_3)$ and (b) $\pi(\hat{q}_1)$. For both plots, the less (red) and larger (blue) fluctuating lines correspond to the cases where the optimal feedback control is performed or not, respectively.

Next, Fig. 4 shows the time evolutions of (a) the syndrome $\pi(\hat{q}_2 - \hat{q}_3)$ and (b) the first oscillator's position $\pi(\hat{q}_1)$. For both plots, the less (red) and larger (blue) fluctuating lines correspond to the cases where the optimal feedback control is performed or not, respectively. The error bars show the standard deviation of the estimation error, which is calculated from the solution to the Riccati equation (19). The parameters are taken as $\mu = -0.4$ and $r = 10^{-9}$, which were shown in Fig. 3 to be the optimal values attaining the maximum fidelity.

Note that, because α_{in} is unknown, \hat{q}_1 cannot be exactly estimated; but we here plot the exact value of $\pi(\hat{q}_1)$ just for demonstration. Figure 4 (a) demonstrates that the fluctuation of the syndrome is certainly suppressed by the feedback control; this is the result that should be expected, since the controller is designed to minimize the syndrome errors. On the other hand, it is not straightforwardly expected that controlling \hat{q}_1 would actually work well, but Fig. 4 (b) demonstrates that the feedback control based on the syndrome estimation is fairly effective for correcting the error acting on each position of the oscillator.

B. Squeezed source state

Here we are concerned with the situation where the source state is squeezed. The squeezing parameter is taken as $\mu_1 = \log(2M_1 + 2N_1 + 1)$ with M_1 real, and N_1 and M_1 are both unknown in addition to α_{in} . We then have to use the filter for \hat{s}_2 , which does not contain N_1 and M_1 . But for comparison we further consider a squeezed source state with known covariance and unknown mean value; in this case the Kalman filter for \hat{s}_1 can be used for feedback control. This comparison will reveal how much the additional information (i.e., the estimate of $\hat{p}_1 + \hat{p}_2 + \hat{p}_3$) improves the control performance at the expense of limiting the class of input states.

Figure 5 shows the fidelity between the input state and the controlled oscillators' steady state, versus the squeezing parameters μ and μ_1 . The control penalty is $r = 10^{-9}$. In the figure the upper and lower surfaces correspond to the filter for \hat{s}_1 and \hat{s}_2 , respectively. First, a notable fact observed from the lower surface is that, when aiming to transfer the input state with the completely unknown source state, the maximum fidelity is attained when $\mu_1 = 0$; that is, squeezing the source state always decreases the fidelity. This is consistent with the standard understanding that an unknown squeezed state is in general fragile to the thermal noise, because it randomly rotates the phase of the state, which as a result brings that state into a mixed state. On the other hand, the upper surface in the figure shows that, when transferring the input state with known fluctuation, which means that \hat{s}_1 can be used for estimation and control, squeezing the source state improves the fidelity. Note that the improvement is observed if $\mu_1 > 0$ and $\mu < 0$; that is, the source state is now momentum-squeezed while the ancilla states are position-squeezed. Actually in this case the input state more closely approximates the GHZ-like state, which is an eigenstate of $\hat{p}_1 + \hat{p}_2 + \hat{p}_3$. Therefore, the additional information $\pi(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)$ should certainly improve the control performance of the state transfer. However, as in the previous case, too much squeezing degrades the fidelity, hence μ_1 must be optimized.

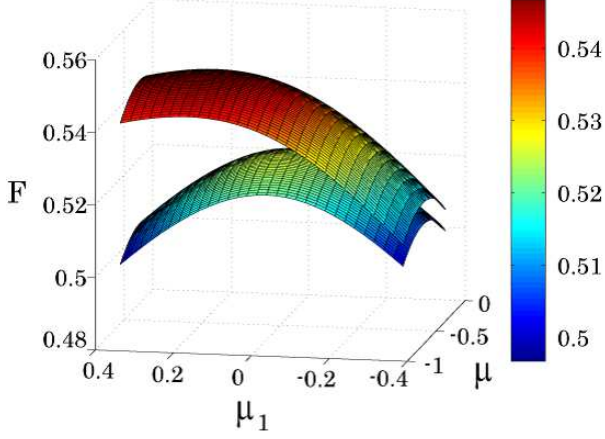


FIG. 5: (Color online) Fidelity between the input state and the controlled oscillators' steady state, versus the squeezing parameters μ and μ_1 . The upper surface corresponds to the case where the estimate $\pi(\hat{s}_1)$ is used to control, while the lower surface does the case with $\pi(\hat{s}_2)$ only available for estimation and control.

VI. CONCLUSION

In this paper we proposed a new feedback control scheme for state transfer, which has the form of QEC.

In particular, due to the Gaussianity of the input state and the linearity of the memory dynamics, the celebrated Kalman filtering and LQG feedback control were employed. We have considered a specific system, but the proposed control strategy is applicable to any linear quantum system, as long as the output process can be properly defined. Although it was shown in the example that the controller is not a very effective one, it is expected that it can show much bigger improvement for some linear systems having relatively weak noise. Also possible combination with a feedforward control scheme [21, 22] could be helpful to attain better performance for Gaussian state transfer.

Appendix A: Covariance matrix of the controlled memory state

The whole closed-loop system of the controlled memory and the estimator is subjected to the following QSDE:

$$d\hat{z} = \begin{pmatrix} A & F \\ \tilde{K}C & \tilde{A} - \tilde{K}C\tilde{B}^\top + \tilde{B}F \end{pmatrix} \hat{z}dt + \begin{pmatrix} B \\ \tilde{K}D \end{pmatrix} d\hat{W},$$

where $\hat{z} = (\hat{x}^\top, \pi(\hat{s})^\top)^\top$. Then the covariance matrix $V_z = \langle \Delta\hat{z}\Delta\hat{z}^\top + (\Delta\hat{z}\Delta\hat{z}^\top)^\top \rangle / 2$ with $\Delta\hat{z} = \hat{z} - \langle \hat{z} \rangle$ changes in time according to the following Lyapunov equation:

$$\frac{d}{dt}V_z = \begin{pmatrix} A & F \\ \tilde{K}C & \tilde{A} - \tilde{K}C\tilde{B}^\top + \tilde{B}F \end{pmatrix} V_z + V_z \begin{pmatrix} A & F \\ \tilde{K}C & \tilde{A} - \tilde{K}C\tilde{B}^\top + \tilde{B}F \end{pmatrix}^\top + \begin{pmatrix} B \\ \tilde{K}D \end{pmatrix} \begin{pmatrix} \Lambda & \\ & (n_T + 1/2)I_6 \end{pmatrix} \begin{pmatrix} B^\top & D^\top \tilde{K}^\top \end{pmatrix}. \quad (\text{A1})$$

Now, the covariance matrix of the controlled memory state, V' , corresponds to the first 6×6 block matrix of V_z . In this case, the steady solution V'_∞ can be explicitly obtained as follows:

$$V'_\infty = -\frac{1}{2}(2A - KC + F\tilde{B})^{-1} \left\{ (A - KC + F\tilde{B}, -F\tilde{B}) \begin{pmatrix} B \\ \tilde{K}D \end{pmatrix} \begin{pmatrix} \Lambda & \\ & (n_T + 1/2)I_6 \end{pmatrix} \begin{pmatrix} B^\top & D^\top \tilde{K}^\top \end{pmatrix} \right. \\ \left. \times \begin{pmatrix} A^\top - C^\top K^\top + \tilde{B}^\top F^\top \\ -\tilde{B}^\top F^\top \end{pmatrix} (A - KC)^{-1} (F\tilde{B} + A)^{-1} + \Gamma(n_T + 1/2)I_6 + \nu T \Lambda T^\top \right\}.$$

Note that as $r \rightarrow 0$ (i.e., cheap control), we have $V'_\infty \rightarrow V'_c(\infty)$, which is the steady solution to Eq. (19). That is, in this limit, the quantum fluctuation of the controlled

memory state is maximally reduced down to the level of estimation error.

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- [54] This can be easily seen in a discrete-variable case; For instance, a three-qubit encoded state $|\phi\rangle = a|000\rangle + b|111\rangle$ changes due to the first bit flip error into $|\phi'_1\rangle = a|100\rangle + b|011\rangle$, hence their inner product is $\langle\phi|\phi'_1\rangle = 0$. Likewise, for the second or the third bit-flipping cases the inner products are zeros as well. Therefore the unconditional state after the error, $\hat{\rho} = (|\phi'_1\rangle\langle\phi'_1| + |\phi'_2\rangle\langle\phi'_2| + |\phi'_3\rangle\langle\phi'_3|)/3$, has no overlap with the encoded state, i.e., $F = \langle\phi|\hat{\rho}|\phi\rangle = 0$. That is, without error correction, encoding process can merely degrades the input-output fidelity.